

# Kubo formulas for relativistic fluids in strong magnetic fields

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## Abstract

Magnetohydrodynamics of strongly magnetized relativistic fluids is derived in the ideal and dissipative cases, taking into account the breaking of spatial symmetries by a quantizing magnetic field. A complete set of transport coefficients, consistent with the Curie and Onsager principles, is derived for thermal conduction, as well as shear and bulk viscosities. It is shown that in the most general case the dissipative function contains five shear viscosities, two bulk viscosities, and three thermal conductivity coefficients. We use Zubarev's non-equilibrium statistical operator method to relate these transport coefficients to correlation functions of the equilibrium theory. The desired relations emerge at linear order in the expansion of the non-equilibrium statistical operator with respect to the gradients of relevant statistical parameters (temperature, chemical potential, and velocity.) The transport coefficients are cast in a form that can be conveniently computed using equilibrium (imaginary-time) infrared Green's functions defined with respect to the equilibrium statistical operator.

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## 1. Introduction

Computation of transport coefficients of relativistic systems is of importance in a variety of fields, including relativistic astrophysics, cosmology, and heavy-ion physics. The kinetic theory based on the Boltzmann equation provides an adequate tool for describing transport in these systems in real time

in the dilute regime. It may be systematically extended to denser systems in cases where one can establish a hierarchy of the timescales governing the system. An alternative to the Boltzmann approach is based on the Kubo method, in which non-equilibrium processes are regarded as the response of the system to an external perturbation, which can be computed from *equilibrium* correlation functions. This method has the advantage that the transport coefficients can be evaluated by using equilibrium Green's functions formulated in imaginary time.

In this paper we shall adopt the correlation-function method developed by Zubarev [1, 2, 3] to derive expressions for transport coefficients of strongly magnetized relativistic fluids. The Zubarev method is based on the concept of a non-equilibrium statistical operator, which is a generalization of the equilibrium Gibbs statistical operator to non-equilibrium states. This approach will enable us to obtain kinetic transport coefficients in terms of correlation functions by expanding the operator in terms of small gradients of the thermodynamical parameters. It is generally assumed that the state of the system can be defined in terms of (space-time dependent) fields of temperature, chemical potential, and momentum — a typical parameter set that is sufficient for a complete description of the system. (The set of *relevant parameters* depends in general on the specific problem under consideration.) The transport coefficients are then obtained from linear perturbations of the non-equilibrium statistical operator around its equilibrium value. In this paper we will be concerned with the situation in which the system is in an external Abelian gauge field, *i.e.*, an electromagnetic field.

The physical motivation for studying relativistic fluids in strong magnetic fields is provided by compact stellar objects (neutron stars). It is now well established that neutron stars feature magnetic fields of order  $10^{12}$ - $10^{13}$  G on average, with a subclass of neutron stars (magnetars) having magnetic fields of order  $10^{15}$ - $10^{16}$  G. Although the *origin* of such large magnetic fields is not known, one possibility is the conservation of magnetic flux [4, 5] under collapse from ordinary stellar dimensions. Alternatively, the field could be generated by a dynamo mechanism [6]. The field strengths  $B \leq 10^{16}$  G inferred for magnetars [7, 8] refer to the surface of a compact star. The internal fields could be by several orders of magnitude larger. An upper limit on the magnitude of the magnetic field  $B \leq 10^{18}$  G is set by the virial theorem for a star in gravitational equilibrium [9, 10]. Observationally, soft gamma-ray repeaters (SGRs) and anomalous x-ray pulsars (AXPs) are now identified with magnetars; these objects have been associated with specific supernova

remnants and form a subclass of young strongly magnetized neutron stars which are distinct from ordinary radio pulsars.

The *equilibrium* properties of dense matter in strong magnetic fields have been studied intensively since the discovery of magnetars. These properties include the equation of state of dense nuclear and quark matter [11, 12, 13], neutrino propagation and pulsar kicks, and neutrino emission and cooling [14, 15, 16, 17, 18, 19, 20, 21]. However, hydrodynamics and non-equilibrium processes in strong fields have not been studied until recently.

In an earlier paper [22], we have developed an anisotropic hydrodynamic theory to describe strongly magnetized strange matter in a compact star. It was shown that the dissipative processes in matter are completely described in terms of seven viscosity coefficients (which include five shear viscosities and two bulk viscosities), three thermal conductivities, and three coefficients of electrical conductivity. Our model is the relativistic counterpart of the non-relativistic formulation of plasma theory in strong magnetic fields due to Braginskii [23].

The purpose of this work is to extend the description of the hydrodynamics of strongly magnetized fluids by relating the hydrodynamic equations to certain correlation functions. The latter can be computed using the methods of equilibrium statistical mechanics. Among several methods to derive Kubo formulas, we choose the one based on the idea of non-equilibrium statistical operators (see Refs. [1, 2, 3] for a detailed discussion of this method for non-magnetized fluids.) Thus, compared to other relativistic treatments of magnetohydrodynamics (see, *e. g.*, Refs. [24]), our approach relates the transport coefficients entering magnetohydrodynamics to retarded correlation functions of the microscopic theory.

This paper is organized as follows. Section 2 is devoted to magnetohydrodynamics of relativistic fluids in the ideal (Subsec. 2.1) and dissipative (Subsec. 2.2) cases. We derive a thermodynamically consistent set of equations describing fluid dynamics in strong magnetic fields and identify the full set of transport coefficients needed for a complete description of dissipation. In Sec. 3 we utilize the non-equilibrium statistical-operator method in the formulation by Zubarev to derive Kubo formulas relating the transport coefficients appearing in our magnetohydrodynamic theory to correlation functions defined in terms of the underlying elementary fields. A discussion of our results and a comparison to previous work is given in Sec. 4. Natural units  $\hbar = k_B = c = 1$  are adopted, and the SI system of units is employed in equations involving electromagnetism. The metric tensor is given

by  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . A number of calculational details are relegated to the appendices.

## 2. Magnetohydrodynamics of relativistic fluids

### 2.1. Non-dissipative fluids

Hydrodynamics arises as a long-wave-length, low-frequency effective theory of matter from an expansion of the energy-momentum tensor  $T^{\mu\nu}$ , the current(s)  $n^\mu$  of conserved charge(s), and the entropy density flux  $s^\mu$ , *etc.*, with respect to (small) gradients of the four-velocity  $u^\mu$  and the thermodynamic parameters of the system, such as the temperature  $T$ , the chemical potential  $\mu$ , *etc.* The zeroth-order terms in this expansion correspond to an ideal fluid, which we will discuss here; the extension to the dissipative case is given in the following subsection.

For the sake of simplicity, we will consider only one conserved charge, *e. g.*, the electric charge. Many physical systems of interest may have other conserved charges, such as baryon number, isospin, *etc.* The extension of our discussion to multiple conserved charges is straightforward. The hydrodynamic equations can be expressed as conservation laws for the energy-momentum tensor  $T^{\mu\nu}$  and the conserved currents, here the electric current  $n^\mu$ ,

$$\partial_\mu T^{\mu\nu} = 0, \quad (1)$$

$$\partial_\mu n^\mu = 0. \quad (2)$$

In the presence of an electromagnetic field the zeroth-order contribution to the energy-momentum tensor can be written as [22, 25, 26]

$$T_0^{\mu\nu} = T_{\text{F0}}^{\mu\nu} + T_{\text{EM}}^{\mu\nu}, \quad (3)$$

$$T_{\text{F0}}^{\mu\nu} = \varepsilon u^\mu u^\nu - P \Delta^{\mu\nu} - \frac{1}{2} (M^{\mu\lambda} F_\lambda{}^\nu + M^{\nu\lambda} F_\lambda{}^\mu), \quad (4)$$

$$T_{\text{EM}}^{\mu\nu} = -F^{\mu\lambda} F_\lambda{}^\nu + \frac{g^{\mu\nu}}{4} F^{\rho\sigma} F_{\rho\sigma}, \quad (5)$$

where  $F^{\mu\nu}$  is the field-strength tensor,  $\varepsilon$  and  $P$  are the local energy density and the thermodynamic pressure,  $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$  is the projector on the directions orthogonal to  $u^\mu$ , and  $M^{\mu\nu}$  is the polarization tensor. In the non-dissipative limit, the entropy is conserved in addition to the charge, *i.e.*, it

obeys a continuity equation analogous to Eq. (2). The charge and entropy currents in the non-dissipative theory are

$$n_0^\mu = n u^\mu, \quad (6)$$

$$s_0^\mu = s u^\mu, \quad (7)$$

where  $n$  and  $s$  are the electric charge and entropy densities measured in the rest frame of the fluid. For the following discussion it will be convenient to decompose the tensor  $F^{\mu\nu}$  into components parallel and perpendicular to  $u^\mu$ ,

$$\begin{aligned} F^{\mu\nu} &= F^{\mu\lambda} u_\lambda u^\nu - F^{\nu\lambda} u_\lambda u^\mu + \Delta^\mu_\alpha F^{\alpha\beta} \Delta_\beta^\nu \\ &\equiv E^\mu u^\nu - E^\nu u^\mu + \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} (u_\alpha B_\beta - u_\beta B_\alpha), \end{aligned} \quad (8)$$

where we have defined the four-vectors  $E^\mu \equiv F^{\mu\nu} u_\nu$  and  $B^\mu \equiv \epsilon^{\mu\nu\alpha\beta} F_{\nu\alpha} u_\beta / 2$ , with  $\epsilon^{\mu\nu\alpha\beta}$  denoting the totally anti-symmetric Levi-Civita tensor. In the rest frame of the fluid,  $u^\mu = (1, \mathbf{0})$ , we find that  $E^0 = B^0 = 0$ ,  $E^i = F^{i0}$  and  $B^i = -\epsilon^{ijk} F_{jk} / 2$ , where the indices  $i, j, k$  run over 1, 2, 3. The latter three vectors are precisely the electric and magnetic fields in this frame. Therefore,  $E^\mu$  and  $B^\mu$  can be interpreted as the electric and magnetic fields measured in the frame in which the fluid moves with a velocity  $u^\mu$ .

The antisymmetric polarization tensor  $M^{\mu\nu}$  describes the response of matter to the applied field strength  $F^{\mu\nu}$ . In the case of systems described by a grand canonical ensemble, it is given by  $M^{\mu\nu} \equiv -\partial\Omega/\partial F_{\mu\nu}$  in terms of the thermodynamic potential  $\Omega$ . For later use, we also define the in-medium field strength tensor  $H^{\mu\nu} \equiv F^{\mu\nu} - M^{\mu\nu}$ . In analogy to what is done for  $F^{\mu\nu}$ , we can decompose  $M^{\mu\nu}$  and  $H^{\mu\nu}$  as

$$M^{\mu\nu} = P^\nu u^\mu - P^\mu u^\nu + \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} (M_\beta u_\alpha - M_\alpha u_\beta), \quad (9)$$

$$H^{\mu\nu} = D^\mu u^\nu - D^\nu u^\mu + \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} (H_\beta u_\alpha - H_\alpha u_\beta), \quad (10)$$

with  $P^\mu \equiv -M^{\mu\nu} u_\nu$ ,  $M^\mu \equiv \epsilon^{\mu\nu\alpha\beta} M_{\nu\alpha} u_\beta / 2$ ,  $H^\mu \equiv \epsilon^{\mu\nu\alpha\beta} H_{\nu\alpha} u_\beta / 2$ , and  $D^\mu \equiv H^{\mu\nu} u_\nu$ .

In the rest frame of the fluid, the non-trivial components of these tensors are  $(F^{10}, F^{20}, F^{30}) = \mathbf{E}$ ,  $(F^{32}, F^{13}, F^{21}) = \mathbf{B}$ ,  $(M^{10}, M^{20}, M^{30}) = -\mathbf{P}$ ,  $(M^{32}, M^{13}, M^{21}) = \mathbf{M}$ ,  $(H^{10}, H^{20}, H^{30}) = \mathbf{D}$ , and  $(H^{32}, H^{13}, H^{21}) = \mathbf{H}$ . Here  $\mathbf{P}$  and  $\mathbf{M}$  are the electric polarization vector and magnetization vector, respectively. In linear approximation they are related to the fields  $\mathbf{E}$

and  $\mathbf{B}$  by the usual expressions  $\mathbf{P} = \chi_e \mathbf{E}$  and  $\mathbf{M} = \chi_m \mathbf{B}$ , where  $\chi_e$  and  $\chi_m$  are the electric and magnetic susceptibilities. Note that the four-vectors  $E^\mu, B^\mu, \dots$  are all space-like, *i.e.*,  $E^\mu u_\mu = 0, B^\mu u_\mu = 0, \dots$ , and are normalized as  $E^\mu E_\mu = -E^2, B^\mu B_\mu = -B^2, \dots$ , with  $E \equiv |\mathbf{E}|$  and  $B \equiv |\mathbf{B}|$ .

For many systems encountered in astrophysics, one example being the interior of a neutron star, the electric field is much weaker than the magnetic field. Therefore, in the following discussion we will omit the contributions originating from the electric field, but will occasionally comment on how our expression will be modified by it. Next let us introduce the four-vector  $b^\mu \equiv B^\mu/B$ , which is normalized as  $b^\mu b_\mu = -1$  (in agreement with the normalization of the vector  $B^\mu$  above), along with the antisymmetric rank-2 tensor  $b^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} b_\alpha u_\beta$ . Then, upon dropping the electric field in the field strength tensors, we write these in the compact form

$$F^{\mu\nu} = -B b^{\mu\nu}, \quad (11)$$

$$M^{\mu\nu} = -M b^{\mu\nu}, \quad (12)$$

$$H^{\mu\nu} = -H b^{\mu\nu}, \quad (13)$$

where  $M \equiv |\mathbf{M}|$  and  $H \equiv |\mathbf{H}|$ . Again neglecting electric fields, the matter and field contributions to the energy-momentum tensor (3) can now be written in terms of  $b^\mu$  and  $b^{\mu\nu}$  as (see, *e.g.*, Refs. [22, 24])

$$T_{F0}^{\mu\nu} = \varepsilon u^\mu u^\nu - P_\perp \Xi^{\mu\nu} + P_\parallel b^\mu b^\nu, \quad (14)$$

$$T_{EM}^{\mu\nu} = \frac{1}{2} B^2 (u^\mu u^\nu - \Xi^{\mu\nu} - b^\mu b^\nu), \quad (15)$$

where  $\Xi^{\mu\nu} \equiv \Delta^{\mu\nu} + b^\mu b^\nu$  is the projection tensor on the direction orthogonal to both  $u^\mu$  and  $b^\mu$ . Here we have defined the transverse and longitudinal pressures  $P_\perp = P - MB$  and  $P_\parallel = P$  relative to the vector  $b^\mu$ . In the absence of a magnetic field, the fluid is isotropic and  $P_\perp = P_\parallel = P$ , where  $P$  is the thermodynamic pressure defined in Eq. (5). In the local rest frame of the fluid, we have  $b^\mu = (0, 0, 0, 1)$ , where without loss of generality the direction of the magnetic field is chosen as the  $z$ -axis. We see then that the electromagnetic tensor takes the usual form, while  $T_{F0}^{\mu\nu} = \text{diag}(\varepsilon, P_\perp, P_\perp, P_\parallel)$ , as required.

Next we would like to check the consistency of the terms that appear in  $T_{F0}^{\mu\nu}$  with the formulas of standard thermodynamics involving electromagnetic fields. By using the thermodynamic relation

$$\varepsilon = Ts + \mu n - P \quad (16)$$

and the conservation equations for  $n_0^\mu$  and  $s_0^\mu$  in ideal hydrodynamics, one can show that the hydrodynamic equation  $u_\nu \partial_\mu T_0^{\mu\nu} = 0$  together with the Maxwell equation (22) implies

$$D\varepsilon = TDs + \mu Dn - MDB, \quad (17)$$

which is consistent with the standard thermodynamic relation

$$d\varepsilon = Tds + \mu dn - MdB. \quad (18)$$

From Eq. (16) and Eq. (18), one can obtain the Gibbs-Duhem relation,

$$dP = sdT + nd\mu + MdB. \quad (19)$$

We should note that the potential energy  $-MB$  has already been included in our definition of  $\varepsilon$ . Otherwise, new terms  $-MB$ ,  $-D(MB)$ , and  $-d(MB)$  should be added to the left-hand sides of Eq. (16), Eq. (17), and Eq. (18), respectively, while the Gibbs-Duhem relation keeps the form of Eq. (19).

The complete set of non-dissipative hydrodynamic equation includes the Maxwell equations, which we state for completeness:

$$\partial_\nu (B^\mu u^\nu - B^\nu u^\mu) = 0, \quad (20)$$

$$\partial_\mu H^{\mu\nu} = n^\nu. \quad (21)$$

Contracting Eq. (20) with  $b_\mu$  gives the so-called induction equation

$$\theta + D \ln B - u^\nu b^\mu \partial_\mu b_\nu = 0, \quad (22)$$

where  $\theta \equiv \partial_\mu u^\mu$  is the velocity divergence and  $D \equiv u^\mu \partial_\mu$  is the substantial (co-moving) time-derivative. (Note that  $b_\mu D b^\mu = b^\mu D b_\mu = 0$ .) In closing this section, we confirm that the form of the energy-momentum tensor  $T_{F0}^{\mu\nu}$  is consistent with standard thermodynamic formulas for matter in electromagnetic fields [22].

## 2.2. Dissipative fluids

The next-to-leading (first-order) derivative expansion of conserved quantities leads to the Navier-Stokes-Fourier-Ohm theory which, as is well known, includes dissipative effects. The continuity equation for the entropy now changes to

$$T \partial_\mu s^\mu \geq 0 \quad (23)$$

(second law of thermodynamics). In the dissipative theory the quantities  $T^{\mu\nu}$ ,  $n^\mu$ , and  $s^\mu$  can be generally expressed as

$$\begin{aligned} T^{\mu\nu} &= T_0^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \tau^{\mu\nu}, \\ n^\mu &= n u^\mu + j^\mu, \\ s^\mu &= s u^\mu + j_s^\mu, \end{aligned} \tag{24}$$

where  $h^\mu$  is the energy flux,  $\tau^{\mu\nu}$  is the viscous stress tensor, and  $j^\mu$  and  $j_s^\mu$  are the charge and entropy diffusion fluxes. They are all orthogonal to  $u^\mu$ , reflecting the fact that the dissipation in the fluid should be spatial. We shall assume that  $j_s^\mu$  can be expressed as a linear combination of  $h^\mu$  and  $j^\mu$ . This allows us to incorporate the fact that the entropy flux is determined by the energy and charge diffusion fluxes. Thus, following Refs. [27, 28], we write

$$j_s^\mu = \gamma h^\mu - \alpha j^\mu, \tag{25}$$

the coefficients  $\gamma$  and  $\alpha$  being functions of thermodynamic variables.

In order to discuss the dissipative parts, let us first define the four-velocity  $u^\mu$ , since it is not unique when one allows for energy exchange by thermal conduction. We employ the Landau-Lifshitz frame, in which  $u^\mu$  is chosen to be parallel to the energy density flow, *i.e.*,  $u_\nu T^{\mu\nu} = u^\mu \varepsilon$ . It follows from the first equation (24) that  $h^\mu = 0$ . We project Eq. (1) onto  $u^\nu$  and exploit the fact that electric field  $E_\lambda$  is zero. Utilizing Eq. (25), straightforward manipulations lead to

$$D\varepsilon + (\varepsilon + P)\theta - \tau^{\mu\nu} \partial_\mu u_\nu + M D B = j^\lambda E_\lambda = 0. \tag{26}$$

Combining this equation with the thermodynamic relation  $\varepsilon = Ts + \mu n - P$  and the continuity equation (2), we arrive at

$$T \partial_\sigma s^\sigma = \tau^{\sigma\nu} w_{\sigma\nu} - j^\sigma T \nabla_\sigma \alpha + (\mu - T\alpha) \partial_\sigma j^\sigma, \tag{27}$$

where  $\nabla_\sigma \equiv \Delta_{\sigma\nu} \partial^\nu$  and  $w^{\sigma\nu} \equiv \frac{1}{2} (\nabla^\sigma u^\nu + \nabla^\nu u^\sigma)$ . The first and second terms on the right-hand side of Eq. (27) (the dissipative function) correspond to viscous and thermal dissipation, respectively, whereas the last term vanishes due to our choice of  $\alpha$ . For a thermodynamically and hydrodynamically stable system the dissipative function should be non-negative. This implies that for small perturbations it must be a quadratic form, so we can write

$$T\alpha = \mu, \tag{28}$$

$$\tau^{\mu\nu} = \eta^{\mu\nu\alpha\beta} w_{\alpha\beta}, \tag{29}$$

$$j^\mu = \kappa^{\mu\nu} T \nabla_\nu \alpha, \tag{30}$$



where  $\eta^{\mu\nu\alpha\beta}$  is the rank-four tensor of viscosity coefficients and  $\kappa^{\mu\nu}$  is the thermal conductivity tensor with respect to the diffusion flux of electric charge density. By definition,  $\eta^{\mu\nu\alpha\beta}$  is symmetric in the pairs of indices  $\alpha, \beta$  and  $\mu, \nu$ . It necessarily satisfies the condition  $\eta^{\mu\nu\alpha\beta}(B^\sigma) = \eta^{\alpha\beta\mu\nu}(-B^\sigma)$ , which is Onsager's symmetry principle for transport coefficients. Similarly, the tensors  $\kappa^{\mu\nu}$  and  $\sigma^{\mu\nu}$  should satisfy the conditions  $\kappa^{\mu\nu}(B^\lambda) = \kappa^{\nu\mu}(-B^\lambda)$  and  $\sigma^{\mu\nu}(B^\lambda) = \sigma^{\nu\mu}(-B^\lambda)$ . Furthermore, all the tensors of transport coefficients  $\eta^{\mu\nu\alpha\beta}$ ,  $\kappa^{\mu\nu}$ , and  $\sigma^{\mu\nu}$  must be orthogonal to  $u^\mu$  by definition.

The constructive equations connecting the irreversible fluxes  $\tau_{\mu\nu}$ ,  $j_\mu$  and the thermodynamic forces  $w_{\mu\nu}$ ,  $\nabla_\mu\alpha$  in dissipative magnetohydrodynamics may have a more general tensor structure

$$\begin{aligned}\tau^{\mu\nu} &= \eta^{\mu\nu\alpha\beta} w_{\alpha\beta} + \lambda^{\mu\nu\rho} T \nabla_\rho \alpha, \\ j^\mu &= \gamma^{\mu\rho\sigma} w_{\rho\sigma} + \kappa^{\mu\nu} T \nabla_\nu \alpha,\end{aligned}\tag{31}$$

where  $\lambda^{\mu\nu\rho}$  and  $\gamma^{\mu\rho\sigma}$  are rank-three tensors representing the coupling between charge diffusion and energy-momentum transport. However, the Curie principle, which states that linear transport relations can be realized between irreducible tensors of the same *rank and parity*, does not allow for such higher-rank tensors  $\lambda^{\mu\nu\rho}$  and  $\gamma^{\mu\rho\sigma}$ . There are, however, some exceptions as, *e. g.*, systems with parity violation [29]. In strong magnetic fields the ranks of the tensors are changed compared to the non-magnetic case because of the breaking of spatial symmetries by the magnetic field. However, the parity of the tensors with respect to reflections should be preserved. We therefore conclude that  $\lambda^{\mu\nu\rho} = 0 = \gamma^{\rho\mu\nu}$ .

The tensors  $\eta^{\mu\nu\alpha\beta}$  and  $\kappa^{\mu\nu}$  can be decomposed into sums of irreducible tensor combinations constructed from the vectors  $u^\mu$ ,  $b^\mu$ ,  $g^{\mu\nu}$ , and  $b^{\mu\nu}$ . In the case of the tensor  $\eta^{\mu\nu\alpha\beta}$ , all the independent irreducible tensor combinations that share its symmetries and are orthogonal to  $u^\mu$  are given by [22]

$$\begin{aligned}\text{(i)} \quad & \Delta^{\mu\nu} \Delta^{\alpha\beta}, \\ \text{(ii)} \quad & \Delta^{\mu\alpha} \Delta^{\nu\beta} + \Delta^{\mu\beta} \Delta^{\nu\alpha}, \\ \text{(iii)} \quad & \Delta^{\mu\nu} b^\alpha b^\beta + \Delta^{\alpha\beta} b^\mu b^\nu, \\ \text{(iv)} \quad & b^\mu b^\nu b^\alpha b^\beta, \\ \text{(v)} \quad & \Delta^{\mu\alpha} b^\nu b^\beta + \Delta^{\nu\beta} b^\mu b^\alpha + \Delta^{\mu\beta} b^\nu b^\alpha + \Delta^{\nu\alpha} b^\mu b^\beta, \\ \text{(vi)} \quad & \Delta^{\mu\alpha} b^\nu b^\beta + \Delta^{\nu\beta} b^\mu b^\alpha + \Delta^{\mu\beta} b^\nu b^\alpha + \Delta^{\nu\alpha} b^\mu b^\beta, \\ \text{(vii)} \quad & b^{\mu\alpha} b^\nu b^\beta + b^{\nu\beta} b^\mu b^\alpha + b^{\mu\beta} b^\nu b^\alpha + b^{\nu\alpha} b^\mu b^\beta.\end{aligned}\tag{32}$$

In Ref. [22] the combination  $b^{\mu\alpha}b^{\nu\beta} + b^{\mu\beta}b^{\nu\alpha}$  was added to the above list. However, as shown in Appendix A, this tensor can be written as a linear combination of tensors listed in Eq. (32); specifically  $b^{\mu\alpha}b^{\nu\beta} + b^{\mu\beta}b^{\nu\alpha} = 2(\text{i}) - (\text{ii}) + 2(\text{iii}) - (\text{v})$ .

The independent irreducible tensor combinations that have the symmetry of  $\kappa^{\mu\nu}$  and are orthogonal to  $u^\mu$  read

$$(\text{viii}) \ \Delta^{\mu\nu}, \quad (\text{ix}) \ b^\mu b^\nu, \quad (\text{x}) \ b^{\mu\nu}. \quad (33)$$

In order to separate the contributions of shear and bulk viscosities we will need to construct tensor structures that differ from those used in Ref. [22]. An appropriate set is given by

$$\begin{aligned} (\text{i}') \quad & \Delta^{\mu\nu} \Delta^{\alpha\beta} = (\text{i}), \\ (\text{ii}') \quad & \Delta^{\mu\alpha} \Delta^{\nu\beta} + \Delta^{\mu\beta} \Delta^{\nu\alpha} = (\text{ii}), \\ (\text{iii}') \quad & \Xi^{\mu\nu} \Xi^{\alpha\beta} = 2(\text{iii}) + 2(\text{i}) + 2(\text{iv}), \\ (\text{iv}') \quad & b^\mu b^\nu b^\alpha b^\beta = (\text{iv}), \\ (\text{v}') \quad & \Xi^{\mu\alpha} b^\nu b^\beta + \Xi^{\nu\beta} b^\mu b^\alpha + \Xi^{\mu\beta} b^\nu b^\alpha + \Xi^{\nu\alpha} b^\mu b^\beta = (\text{v}) + 4(\text{iv}), \\ (\text{vi}') \quad & \Xi^{\mu\alpha} b^{\nu\beta} + \Xi^{\nu\beta} b^{\mu\alpha} + \Xi^{\mu\beta} b^{\nu\alpha} + \Xi^{\nu\alpha} b^{\mu\beta} = (\text{vi}) + (\text{vii}), \\ (\text{vii}') \quad & b^{\mu\alpha} b^{\nu\beta} + b^{\nu\beta} b^{\mu\alpha} + b^{\mu\beta} b^{\nu\alpha} + b^{\nu\alpha} b^{\mu\beta} = (\text{vii}). \end{aligned} \quad (34)$$

From these tensors we construct the following combinations, which are either traceless,

$$\begin{aligned} (\text{i}'') \quad & \Delta^{\mu\alpha} \Delta^{\nu\beta} + \Delta^{\mu\beta} \Delta^{\nu\alpha} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta}, \\ (\text{ii}'') \quad & \left( \Delta^{\mu\nu} - \frac{3}{2} \Xi^{\mu\nu} \right) \left( \Delta^{\alpha\beta} - \frac{3}{2} \Xi^{\alpha\beta} \right), \\ (\text{iii}'') \quad & -\Xi^{\mu\alpha} b^\nu b^\beta - \Xi^{\nu\beta} b^\mu b^\alpha - \Xi^{\mu\beta} b^\nu b^\alpha - \Xi^{\nu\alpha} b^\mu b^\beta, \\ (\text{iv}'') \quad & -\Xi^{\mu\alpha} b^{\nu\beta} - \Xi^{\nu\beta} b^{\mu\alpha} - \Xi^{\mu\beta} b^{\nu\alpha} - \Xi^{\nu\alpha} b^{\mu\beta}, \\ (\text{v}'') \quad & b^{\mu\alpha} b^{\nu\beta} + b^{\nu\beta} b^{\mu\alpha} + b^{\mu\beta} b^{\nu\alpha} + b^{\nu\alpha} b^{\mu\beta}, \end{aligned} \quad (35)$$

or have non-vanishing trace,

$$\begin{aligned} (\text{vi}'') \quad & \frac{3}{2} \Xi^{\mu\nu} \Xi^{\alpha\beta}, \\ (\text{vii}'') \quad & 3b^\mu b^\nu b^\alpha b^\beta. \end{aligned} \quad (36)$$

Now we can write down the most general expression for  $\eta^{\mu\nu\alpha\beta}$  as

$$\eta^{\mu\nu\alpha\beta} = \eta_0(\text{i}''') + \eta_1(\text{ii}''') + \eta_2(\text{iii}''') + \eta_3(\text{iv}''') + \eta_4(\text{v}''') + \zeta_\perp(\text{vi}''') + \zeta_\parallel(\text{vii}'''). \quad (37)$$

Similarly, we also have the general decompositions for  $\kappa^{\mu\nu}$ , namely

$$\kappa^{\mu\nu} = \kappa_\perp \Xi^{\mu\nu} - \kappa_\parallel b^\mu b^\nu - \kappa_\times b^{\mu\nu}. \quad (38)$$

Thus we conclude that a fluid in a magnetic field has in general seven independent viscosity coefficients and three independent thermal conduction coefficients, which are defined through Eq. (37) and Eq. (38), respectively. Substituting Eqs. (37) and (38) into Eqs. (29) and (30), respectively, we obtain the expressions for the viscous stress tensor and thermal flux in terms of viscosity coefficients and thermal conductivities,

$$\begin{aligned} \tau^{\mu\nu} &= 2\eta_0 \left( w^{\mu\nu} - \frac{1}{3} \Delta^{\mu\nu} \theta \right) + \eta_1 \left( \Delta^{\mu\nu} - \frac{3}{2} \Xi^{\mu\nu} \right) \left( \theta - \frac{3}{2} \phi \right) \\ &- 2\eta_2 (b^\mu \Xi^{\nu\alpha} b^\beta + b^\nu \Xi^{\mu\alpha} b^\beta) w_{\alpha\beta} - 2\eta_3 (\Xi^{\mu\alpha} b^{\nu\beta} + \Xi^{\nu\alpha} b^{\mu\beta}) w_{\alpha\beta} \\ &+ 2\eta_4 (b^{\mu\alpha} b^\nu b^\beta + b^{\nu\alpha} b^\mu b^\beta) w_{\alpha\beta} + \frac{3}{2} \zeta_\perp \Xi^{\mu\nu} \phi + 3\zeta_\parallel b^\mu b^\nu \varphi, \end{aligned} \quad (39)$$

$$j^\mu = \kappa_\perp T \Xi^{\mu\nu} \nabla_\nu \alpha - \kappa_\parallel b^\mu b^\nu T \nabla_\nu \alpha - \kappa_\times b^{\mu\nu} T \nabla_\nu \alpha, \quad (40)$$

where  $\phi \equiv \Xi^{\mu\nu} w_{\mu\nu}$  and  $\varphi \equiv b^\mu b^\nu w_{\mu\nu}$ . The coefficients of the tensor  $\tau^{\mu\nu}$  multiplying traceless tensors are identified as shear viscosities, while the coefficients multiplying the tensors with non-vanishing trace are identified as bulk viscosities. The  $\kappa$ 's are the thermal conductivities.

The divergence of the entropy-density flux (27) can now be written explicitly as

$$\begin{aligned} T \partial_\mu s^\mu &= 2\eta_0 \left( w^{\mu\nu} - \frac{1}{3} \Delta^{\mu\nu} \theta \right) \left( w_{\mu\nu} - \frac{1}{3} \Delta_{\mu\nu} \theta \right) + \eta_1 \left( \theta - \frac{3}{2} \phi \right)^2 \\ &+ 2\eta_2 (b^\mu b_\rho w^{\rho\nu} - b^\nu b_\rho w^{\rho\mu}) (b_\mu b^\rho w_{\rho\nu} - b_\nu b^\rho w_{\rho\mu}) \\ &+ \frac{3}{2} \zeta_\perp \phi^2 + 3\zeta_\parallel \varphi^2 - \kappa_\perp T^2 \Xi^{\mu\nu} \nabla_\nu \alpha \nabla_\mu \alpha + \kappa_\parallel T^2 (b^\mu \nabla_\mu \alpha)^2. \end{aligned} \quad (41)$$

One should note that the terms corresponding to the transport coefficients  $\eta_3, \eta_4$ , and  $\kappa_\times$  in Eqs. (39)-(40) do not contribute to the divergence of the entropy-density flux. For stable systems, all the other transport coefficients

must be positive to satisfy the second law of thermodynamics (positivity of the dissipative function.) There are, however, examples in which this condition may be violated. Indeed, as shown in Ref. [22], strange quark matter (if it occurs in some form in compact stars) may have negative transverse bulk viscosity  $\zeta_{\perp}$  in strong magnetic fields, in situations where non-leptonic weak processes are dominant. The change in the sign of this transport coefficient can be linked to the onset of a hydrodynamic instability [22].

Having determined the general form of the relativistic hydrodynamic equations in strong magnetic fields, we turn now to the problem of determining the transport coefficients entering these equations. We have previously determined the bulk viscosity [22] in the specific case of strange matter, where knowledge of the equation of state and perturbative weak interactions rates is sufficient to fix the value of  $\zeta$ . In order to obtain the remaining transport coefficients one must resort to the microscopic theory. Two general approaches are available for this task: one is based on the solution of a (linearized) Boltzmann equation; the second is based on the evaluation of certain correlation functions. The first approach is suitable for dilute systems or systems with well-defined quasiparticles. The second approach is more general, since the required correlation functions can in principle be evaluated for a system with arbitrarily strong interactions far from equilibrium. We will follow the second approach to express the transport coefficients of a fluid in a strong magnetic field in terms correlation functions. In order to do so, we apply the powerful method of non-equilibrium statistical operators developed by Zubarev [1].

### 3. Non-equilibrium Statistical Operators and Kubo Formulas

Zubarev's method of non-equilibrium statistical operators (NESO) allows one to develop a statistical theory of irreversible processes via a generalization of the Gibbs approach to equilibrium systems. Zubarev's method assumes that one can construct a statistical ensemble representing the macroscopic state of the system in non-equilibrium. In general this is possible if one is interested in quantities that are averaged over time intervals that are long enough that the initial state of the system is unimportant and the number of parameters describing the system is correspondingly reduced.

We consider the system to be in the hydrodynamic regime (in which the thermodynamical parameters such as temperature and chemical potentials can be defined locally) and, following Zubarev, choose the NESO in the

form [1]

$$\hat{\rho}(t) = Q^{-1} \exp \left[ - \int d^3 \mathbf{x} \hat{Z}(\mathbf{x}, t) \right], \quad (42)$$

$$Q = \text{Tr} \exp \left[ - \int d^3 \mathbf{x} \hat{Z}(\mathbf{x}, t) \right], \quad (43)$$

where the operator  $\hat{Z}$  reads

$$\hat{Z}(\mathbf{x}, t) = \varepsilon \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)} \left[ \beta^\nu(\mathbf{x}, t_1) \hat{T}_{0\nu}(\mathbf{x}, t_1) - \alpha(\mathbf{x}, t_1) \hat{N}^0(\mathbf{x}, t_1) \right], \quad (44)$$

with  $\varepsilon \rightarrow +0$  after the thermodynamic limit is taken, and

$$\beta^\nu(\mathbf{x}, t) = \beta(\mathbf{x}, t) u^\nu(\mathbf{x}, t), \quad (45)$$

$$\alpha(\mathbf{x}, t) = \beta(\mathbf{x}, t) \mu(\mathbf{x}, t). \quad (46)$$

Physically the parameters  $\beta$ ,  $\mu$ , and  $u^\mu$  stand for the inverse local equilibrium temperature, chemical potential, and flow four-velocity, respectively. The conditions needed to make such identifications will be addressed below. The operators of the energy-momentum tensor  $\hat{T}^{\mu\nu}$  and the electric current  $\hat{N}^\mu$  satisfy the local conservation laws

$$\partial_\mu \hat{T}^{\mu\nu} = 0, \quad \partial_\mu \hat{N}^\mu = 0. \quad (47)$$

Integrating Eq. (44) by parts and using the conservation laws (47), we rewrite the exponent of Eq. (42) as

$$\begin{aligned} \int d^3 \mathbf{x} \hat{Z}(\mathbf{x}, t) &= \int d^3 \mathbf{x} \left[ \beta^\nu(\mathbf{x}, t) \hat{T}_{0\nu}(\mathbf{x}, t) - \alpha(\mathbf{x}, t) \hat{N}^0(\mathbf{x}, t) \right] \\ &\quad - \int d^3 \mathbf{x} \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)} \left[ \hat{T}_{\mu\nu}(\mathbf{x}, t_1) \partial^\mu \beta^\nu(\mathbf{x}, t_1) \right. \\ &\quad \left. - \hat{N}^\mu(\mathbf{x}, t_1) \partial_\mu \alpha(\mathbf{x}, t_1) \right]. \end{aligned} \quad (48)$$

The quantities  $\partial^\mu \beta^\nu$ ,  $\partial_\mu \alpha$  are thermodynamic “forces” as they involve gradients of velocity field, temperature, and chemical potential. It is natural to identify the first integral in Eq. (48) as the equilibrium part of the statistical operator and the second as the non-equilibrium part of the statistical operator (the latter vanishing when the space-time variations of the parameters

$\alpha(\mathbf{x}, t)$  and  $\beta^\nu(\mathbf{x}, t)$  vanish.) Thus we may define the *local equilibrium part* of the statistical operator as

$$\hat{\rho}_l(t) = Q_l^{-1} \exp \left[ - \int d^3\mathbf{x} \left( \beta^\nu(\mathbf{x}, t) \hat{T}_{0\nu}(\mathbf{x}, t) - \alpha(\mathbf{x}, t) \hat{N}^0(\mathbf{x}, t) \right) \right], \quad (49)$$

$$Q_l = \text{Tr} \exp \left[ - \int d^3\mathbf{x} \left( \beta^\nu(\mathbf{x}, t) \hat{T}_{0\nu}(\mathbf{x}, t) - \alpha(\mathbf{x}, t) \hat{N}^0(\mathbf{x}, t) \right) \right], \quad (50)$$

and write the complete statistical operator as

$$\hat{\rho} = Q^{-1} e^{-\hat{A} + \hat{B}}, \quad (51)$$

$$Q = \text{Tr} e^{-\hat{A} + \hat{B}}, \quad (52)$$

having introduced the following short-hand notation for the integrals

$$\hat{A}(t) = \int d^3\mathbf{x} \left( \beta^\nu(\mathbf{x}, t) \hat{T}_{0\nu}(\mathbf{x}, t) - \alpha(\mathbf{x}, t) \hat{N}^0(\mathbf{x}, t) \right), \quad (53)$$

$$\hat{B}(t) = \int d^3\mathbf{x} \int_{-\infty}^t dt_1 e^{\varepsilon(t_1 - t)} \hat{C}(\mathbf{x}, t_1), \quad (54)$$

with

$$\hat{C}(\mathbf{x}, t) = \hat{T}_{\mu\nu}(\mathbf{x}, t) \partial^\mu \beta^\nu(\mathbf{x}, t) - \hat{N}^\mu(\mathbf{x}, t) \partial_\mu \alpha(\mathbf{x}, t). \quad (55)$$

Now we consider the case of small perturbations from the equilibrium state, *i.e.*, the case in which the thermodynamic forces are sufficiently small that the time-scales for relaxation of non-equilibrium states are large. In this regime, the relations between the thermodynamic forces and irreversible currents are linear, so that we can approximate the non-equilibrium statistical operator by keeping only the linear term in the expansion of the exponent in Eq. (51). Thus, in the linear approximation the complete statistical operator reads

$$\hat{\rho} = \left[ 1 + \int_0^1 d\tau \left( e^{-\hat{A}\tau} \hat{B} e^{\hat{A}\tau} - \langle \hat{B} \rangle_l \right) \right] \hat{\rho}_l, \quad (56)$$

where  $\langle \hat{B} \rangle_l = \text{Tr} \left( \hat{\rho}_l \hat{B} \right)$  is the average over the local equilibrium operator.

Now we are in a position to evaluate the energy-momentum tensor averaged over the non-equilibrium distribution. We obtain

$$\begin{aligned} \langle \hat{T}^{\mu\nu}(\mathbf{x}, t) \rangle &\equiv \text{Tr} \left[ \rho(t) \hat{T}^{\mu\nu}(\mathbf{x}, t) \right] \\ &= \langle \hat{T}^{\mu\nu}(\mathbf{x}, t) \rangle_l + \delta \langle \hat{T}^{\mu\nu}(\mathbf{x}, t) \rangle, \end{aligned} \quad (57)$$

where

$$\begin{aligned}\delta\langle\hat{T}^{\mu\nu}(\mathbf{x}, t)\rangle &= \text{Tr} \left[ \int_0^1 d\tau e^{-\hat{A}\tau} \hat{B} e^{\hat{A}\tau} \hat{\rho}_l(t) \hat{T}^{\mu\nu}(\mathbf{x}, t) \right] \\ &- \text{Tr} \left[ \langle\hat{B}\rangle_l \hat{\rho}_l(t) \hat{T}^{\mu\nu}(\mathbf{x}, t) \right]\end{aligned}\quad (58)$$

is the deviation from the average equilibrium value of the energy-momentum tensor  $\langle\hat{T}^{\mu\nu}(\mathbf{x}, t)\rangle_l$ . Substituting the integral (54) into Eq. (58), we obtain

$$\begin{aligned}\delta\langle\hat{T}^{\mu\nu}(\mathbf{x}, t)\rangle &= \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)} \\ &\int_0^1 d\tau \langle\hat{T}^{\mu\nu}(\mathbf{x}, t) \left[ e^{-\hat{A}\tau} \hat{C}(\mathbf{x}_1, t_1) e^{\hat{A}\tau} - \langle\hat{C}(\mathbf{x}_1, t_1)\rangle_l \right]\rangle_l.\end{aligned}\quad (59)$$

The notation can be compactified by introducing Kubo correlation functions defined as

$$\left( \hat{X}(\mathbf{x}, t), \hat{Y}(\mathbf{x}', t') \right) \equiv \int_0^1 d\tau \langle \hat{X}(\mathbf{x}, t) \left[ e^{-\hat{A}\tau} \hat{Y}(\mathbf{x}', t') e^{\hat{A}\tau} - \langle \hat{Y}(\mathbf{x}', t') \rangle_l \right] \rangle_l, \quad (60)$$

in terms of which we finally obtain

$$\tau^{\mu\nu} = \delta\langle\hat{T}^{\mu\nu}(\mathbf{x}, t)\rangle = \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)} \left( \hat{T}^{\mu\nu}(\mathbf{x}, t), \hat{C}(\mathbf{x}_1, t_1) \right). \quad (61)$$

Similarly, the linear response of the charge current may be written as

$$j^\mu = \delta\langle\hat{N}^\mu(\mathbf{x}, t)\rangle = \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)} \left( \hat{N}^\mu(\mathbf{x}, t), \hat{C}(\mathbf{x}_1, t_1) \right). \quad (62)$$

Equations (61) and (62) describe the response of the fluid to small perturbations.

In the next step we need to match the averages of the conserved quantities (61) and (62) to those that appear in the hydrodynamic equations. For this purpose, we must first specify the parameters  $\beta$  and  $\mu$  as the inverse temperature and the chemical potential in local equilibrium, respectively. In order to enforce such matching it is sufficient to require that in the rest frame of the fluid [1, 3, 30, 31],

$$\langle\hat{T}^{00}(\mathbf{x}, t)\rangle = \langle\hat{T}^{00}(\mathbf{x}, t)\rangle_l, \quad (63)$$

$$\langle\hat{N}^0(\mathbf{x}, t)\rangle = \langle\hat{N}^0(\mathbf{x}, t)\rangle_l, \quad (64)$$

or in a covariant form  $u^\mu \delta \langle \hat{T}_{\mu\nu} \rangle u^\nu = 0$  and  $u^\mu \delta \langle \hat{N}_\mu \rangle = 0$ . Secondly, we need to identify the parameter  $u^\mu$  with the four-velocity appearing in our hydrodynamic equations in Sec. 2. In general, such identification requires a specific frame, *e.g.*, the Landau-Lifshitz (LL) frame, where the four-velocity is parallel to the energy flow, or the Eckart frame where the four-velocity is parallel to the charge flow. From Eqs. (61)-(62) we see that in the frame where  $v^i = 0$ , neither  $\delta \langle \hat{T}^{0i} \rangle$  nor  $\delta \langle \hat{N}^i \rangle$  necessarily vanishes, *i.e.*, we require a transformation to either the LL or Eckart frame. We choose to work in the LL frame, in which case the three-velocity should be boosted as  $v^i = \delta \langle \hat{T}^{0i} \rangle / (\varepsilon + P)$ , where  $\varepsilon = \langle \hat{T}^{00} \rangle_l$ ,  $P = \sum_i \langle \hat{T}^{ii} \rangle_l / 3$ , and  $n = \langle \hat{N}^0 \rangle_l$  are the equilibrium values of energy density, pressure, and charge density, respectively. (In the Eckart frame the boost is given by  $v^i = \delta \langle \hat{N}^i \rangle / n$ .)

Written in the LL frame, the hydrodynamic stress tensor and electric current are [3]

$$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle_l + \delta \langle \hat{T}^{\mu\nu} \rangle - u^\mu u^\rho \delta \langle \hat{T}_{\rho\sigma} \rangle \Delta^{\sigma\nu} - u^\nu u^\rho \delta \langle \hat{T}_{\rho\sigma} \rangle \Delta^{\sigma\mu}, \quad (65)$$

$$n^\mu = \langle \hat{N}^\mu \rangle_l + \delta \langle \hat{N}^\mu \rangle - \frac{n}{\varepsilon + P} u^\rho \delta \langle \hat{T}_{\rho\sigma} \rangle \Delta^{\sigma\mu}, \quad (66)$$

where the quantities on the left are those appearing in the hydrodynamic discussion of Sec. 2. According to Eqs. (63) and (64) we have  $u^\mu \delta \langle \hat{T}_{\mu\nu} \rangle u^\nu = 0$  and  $u^\mu \delta \langle \hat{N}_\mu \rangle = 0$ , so any linear combination of these expressions can be freely added to the right-hand side of Eq. (65). This allows us to rewrite  $T^{\mu\nu}$  and  $n^\mu$  as

$$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle_l + \delta \langle \hat{K}^{\mu\nu} \rangle, \quad (67)$$

$$n^\mu = \langle \hat{N}^\mu \rangle_l + \delta \langle \hat{G}^\mu \rangle \quad (68)$$

after introducing

$$\begin{aligned} \hat{K}_{\mu\nu} &= \hat{T}^{\rho\sigma} (\Delta_{\rho\mu} \Delta_{\sigma\nu} + \Theta_\beta u_\rho u_\sigma \Delta_{\mu\nu} + \Phi_\beta u_\rho u_\sigma \Xi_{\mu\nu}) \\ &+ \hat{N}^\rho u_\rho (\Theta_\alpha \Delta_{\mu\nu} + \Phi_\alpha \Xi_{\mu\nu}), \end{aligned} \quad (69)$$

$$\hat{G}_\mu = \frac{n}{\varepsilon + P} u_\rho \hat{T}^{\rho\nu} \Delta_{\nu\mu} - \hat{N}^\nu \Delta_{\nu\mu}, \quad (70)$$

where  $\Theta_\beta$ ,  $\Theta_\alpha$ ,  $\Phi_\beta$ , and  $\Phi_\alpha$  are defined in Appendix B.

The advantage of casting  $\hat{T}^{\mu\nu}$  and  $\hat{N}^\mu$  in the form given by Eqs. (67) and (68) is that the operator  $\hat{C}$  can be decomposed as (see Appendix B)

$$\hat{C} = \hat{G}_\rho \nabla^\rho \alpha + \beta \hat{K}_{\rho\sigma} w^{\rho\sigma}, \quad (71)$$



where  $\hat{G}_\rho$  has odd spatial parity and  $\hat{K}_{\rho\sigma}$  has even spatial parity. Later it will be seen that such a decomposition permits us to write the transport coefficients in a symmetric manner.

For further convenience, we decompose the energy-momentum tensor  $\hat{T}^{\mu\nu}$  and the current  $\hat{N}^\mu$  as follows:

$$\begin{aligned}\hat{T}^{\mu\nu} &= \hat{\varepsilon}u^\mu u^\nu - \hat{P}_\perp \Xi^{\mu\nu} + \hat{P}_\parallel b^\mu b^\nu + \hat{h}^\mu u^\nu + \hat{h}^\nu u^\mu - \hat{R}^\mu b^\nu - \hat{R}^\nu b^\mu + \hat{\pi}^{\mu\nu}, \\ \hat{N}^\mu &= \hat{n}u^\mu + \hat{J}^\mu + \hat{l}b^\mu,\end{aligned}\tag{72}$$

the expansion coefficients being defined by

$$\begin{aligned}\hat{\varepsilon} &\equiv u_\mu u_\nu \hat{T}^{\mu\nu}, \\ \hat{P}_\perp &\equiv -\frac{1}{2}\Xi_{\mu\nu}\hat{T}^{\mu\nu}, \\ \hat{P}_\parallel &\equiv b_\mu b_\nu \hat{T}^{\mu\nu}, \\ \hat{h}^\mu &\equiv \Delta^{\mu\rho}\hat{T}_{\rho\sigma}u^\sigma, \\ \hat{R}^\mu &\equiv \Xi^{\mu\rho}\hat{T}_{\rho\sigma}b^\sigma, \\ \hat{\pi}^{\mu\nu} &\equiv \left(\Xi^{\rho\mu}\Xi^{\sigma\nu} - \frac{1}{2}\Xi^{\rho\sigma}\Xi^{\mu\nu}\right)\hat{T}_{\rho\sigma}, \\ \hat{n} &\equiv \hat{N}^\mu u_\mu, \\ \hat{J}^\mu &\equiv \hat{N}_\nu \Xi^{\mu\nu}, \\ \hat{l} &\equiv -\hat{N}^\mu b_\mu.\end{aligned}\tag{73}$$

The operators  $\hat{h}^\mu$ ,  $\hat{R}^\mu$ ,  $\hat{\pi}^{\mu\nu}$ , and  $\hat{J}^\mu$  are all perpendicular to  $u_\mu$ , while  $\hat{R}^\mu$ ,  $\hat{\pi}^{\mu\nu}$ , and  $\hat{J}^\mu$  are also perpendicular to  $b_\mu$ . The physical meanings of these operators are obvious. With these definitions we finally obtain

$$\hat{K}_{\mu\nu} = \hat{\pi}^{\mu\nu} - \tilde{P}_\perp \Xi^{\mu\nu} + \tilde{P}_\parallel b^\mu b^\nu - \hat{R}^\mu b^\nu - \hat{R}^\nu b^\mu,\tag{74}$$

$$\hat{G}_\mu = \frac{n}{\varepsilon + P}h_\mu - \hat{J}_\mu - \hat{l}b_\mu,\tag{75}$$

with  $\tilde{P}_\perp \equiv \hat{P}_\perp - (\Theta_\beta + \Phi_\beta)\hat{\varepsilon} - (\Theta_\alpha + \Phi_\alpha)\hat{n}$ , and  $\tilde{P}_\parallel \equiv \hat{P}_\parallel - \Theta_\beta\hat{\varepsilon} - \Theta_\alpha\hat{n}$ .

The dissipative terms can now be written as correlation functions constructed from Eqs. (74) and (75),

$$\tau^{\mu\nu}(\mathbf{x}, t) = \delta\langle\hat{T}^{\mu\nu}(\mathbf{x}, t)\rangle = \beta \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)}$$

$$\times \left( \hat{K}_{\mu\nu}(\mathbf{x}, t), \hat{K}^{\rho\sigma}(\mathbf{x}_1, t_1) \right) w_{\rho\sigma}(\mathbf{x}_1, t_1), \quad (76)$$

$$\begin{aligned} j^\mu(\mathbf{x}, t) &= \delta \langle \hat{N}^\mu(\mathbf{x}, t) \rangle \\ &= - \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)} \left( \hat{G}^\mu(\mathbf{x}, t), \hat{G}^\rho(\mathbf{x}_1, t_1) \right) \nabla_\rho \alpha(\mathbf{x}_1, t_1), \end{aligned} \quad (77)$$

where we have invoked Curie's principle that the correlators between operators with different spatial parity must vanish.

Now suppose that the changes of the thermodynamical forces are sufficiently small over the correlation lengths of these expressions, such that they can be factored out from the integral. Then we obtain the desired linear relations between the thermodynamical forces and the irreversible currents. The coefficients in these linear relations are easily identified as the viscosities and thermal conductivities:

$$\eta^{\mu\nu\rho\sigma} = \beta \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)} \left( \hat{K}^{\mu\nu}(\mathbf{x}, t), \hat{K}^{\rho\sigma}(\mathbf{x}_1, t_1) \right), \quad (78)$$

$$\kappa^{\mu\nu} = -\beta \int d^3\mathbf{x}_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)} \left( \hat{G}^\mu(\mathbf{x}, t), \hat{G}^\nu(\mathbf{x}_1, t_1) \right). \quad (79)$$

For practical purposes, it is more convenient to express these transport coefficients in terms of retarded Green's functions, which can be computed using the methods of equilibrium Green's functions. Straightforward manipulations lead us (see Appendix Appendix C for details) to

$$\eta^{\mu\nu\rho\sigma} = i \frac{\partial}{\partial \omega} G_\eta^{\mu\nu\rho\sigma}(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \quad (80)$$

$$\kappa^{\mu\nu} = -i \frac{\partial}{\partial \omega} G_\kappa^{\mu\nu}(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \quad (81)$$

where  $G_\eta^{\mu\nu\rho\sigma}(\mathbf{k}, \omega)$  and  $G_\kappa^{\mu\nu}(\mathbf{k}, \omega)$  are the Fourier transforms of the retarded Green's functions

$$\begin{aligned} G_\eta^{\mu\nu\rho\sigma}(\mathbf{x}, t) &\equiv -i\theta(t) \left[ \hat{K}^{\mu\nu}(\mathbf{x}, t), \hat{K}^{\rho\sigma}(\mathbf{0}, 0) \right], \\ G_\kappa^{\mu\nu}(\mathbf{x}, t) &\equiv -i\theta(t) \left[ \hat{G}^\mu(\mathbf{x}, t), \hat{G}^\nu(\mathbf{0}, 0) \right]. \end{aligned} \quad (82)$$

(Note that we are using the same notation for Green's functions in real space-time and for their Fourier transforms in momentum-energy space.) Next, we

match Eq. (80) with Eq. (37), which leads us to

$$\begin{aligned}
\Delta_{\mu\nu}\eta^{\mu\nu\alpha\beta}\Xi_{\alpha\beta} &= 6\zeta_{\perp}, \\
\Delta_{\mu\nu}\eta^{\mu\nu\alpha\beta}b_{\alpha}b_{\beta} &= -3\zeta_{\parallel}, \\
\Delta_{\mu\alpha}\eta^{\mu\nu\alpha\beta}\Delta_{\nu\beta} &= 10\eta_0 + \frac{3}{2}\eta_1 + 4\eta_2 + 3\zeta_{\perp} + 3\zeta_{\parallel}, \\
\Delta_{\mu\alpha}\eta^{\mu\nu\alpha\beta}b_{\nu}b_{\beta} &= -\frac{10}{3}\eta_0 - \eta_1 - 2\eta_2 - 3\zeta_{\parallel}, \\
b_{\mu}b_{\nu}b_{\alpha}b_{\beta}\eta^{\mu\nu\alpha\beta} &= \frac{4}{3}\eta_0 + \eta_1 + 3\zeta_{\parallel}, \\
b_{\mu\alpha}\Xi_{\nu\beta}\eta^{\mu\nu\alpha\beta} &= -8\eta_3, \\
b_{\mu\alpha}b_{\nu}b_{\beta}\eta^{\mu\nu\alpha\beta} &= 2\eta_4.
\end{aligned} \tag{83}$$

These equations uniquely determine the viscosity coefficients; explicitly, they are given by

$$\begin{aligned}
\zeta_{\perp} &= \frac{1}{6}\Delta_{\mu\nu}\eta^{\mu\nu\alpha\beta}\Xi_{\alpha\beta}, \\
\zeta_{\parallel} &= -\frac{1}{3}\Delta_{\mu\nu}\eta^{\mu\nu\alpha\beta}b_{\alpha}b_{\beta}, \\
\eta_0 &= \frac{1}{8}(-\Delta_{\mu\nu}\Delta_{\alpha\beta} - 2\Delta_{\mu\nu}b_{\alpha}b_{\beta} + 2\Delta_{\mu\alpha}\Delta_{\nu\beta} + 4\Delta_{\mu\alpha}b_{\nu}b_{\beta} + b_{\mu}b_{\nu}b_{\alpha}b_{\beta})\eta^{\mu\nu\alpha\beta}, \\
\eta_1 &= \frac{1}{6}(\Delta_{\mu\nu}\Delta_{\alpha\beta} + 8\Delta_{\mu\nu}b_{\alpha}b_{\beta} - 2\Delta_{\mu\alpha}\Delta_{\nu\beta} - 4\Delta_{\mu\alpha}b_{\nu}b_{\beta} + 5b_{\mu}b_{\nu}b_{\alpha}b_{\beta})\eta^{\mu\nu\alpha\beta}, \\
\eta_2 &= \frac{1}{8}(\Delta_{\mu\nu}\Delta_{\alpha\beta} + 2\Delta_{\mu\nu}b_{\alpha}b_{\beta} - 2\Delta_{\mu\alpha}\Delta_{\nu\beta} - 8\Delta_{\mu\alpha}b_{\nu}b_{\beta} - 5b_{\mu}b_{\nu}b_{\alpha}b_{\beta})\eta^{\mu\nu\alpha\beta}, \\
\eta_3 &= -\frac{1}{8}b_{\mu\alpha}\Xi_{\nu\beta}\eta^{\mu\nu\alpha\beta}, \\
\eta_4 &= \frac{1}{2}b_{\mu\alpha}b_{\nu}b_{\beta}\eta^{\mu\nu\alpha\beta}.
\end{aligned} \tag{84}$$

We obtain the thermal conductivities in similar fashion by matching Eq. (81) with Eq. (38). Their explicit form is given by

$$\begin{aligned}
\kappa_{\perp} &= \frac{1}{2}\Xi_{\mu\sigma}\kappa^{\mu\sigma}, \\
\kappa_{\parallel} &= -b_{\mu}b_{\sigma}\kappa^{\mu\sigma}, \\
\kappa_{\times} &= -\frac{1}{2}b_{\mu\sigma}\kappa^{\mu\sigma}.
\end{aligned} \tag{85}$$

Finally, we wish to express the transport coefficients in terms of the Green's functions. Substituting the results for the viscosity and thermal conductivity tensors, Eqs. (80) and (81), into Eqs. (84) and (85), we arrive at

$$\begin{aligned}
\zeta_{\perp} &= -\frac{1}{3} \frac{\partial}{\partial \omega} \left[ 2G_{\tilde{P}_{\perp} \tilde{P}_{\perp}}^R(\mathbf{0}, \omega) + G_{\tilde{P}_{\parallel} \tilde{P}_{\perp}}^R(\mathbf{0}, \omega) \right] \Big|_{\omega \rightarrow 0}, \\
\zeta_{\parallel} &= -\frac{1}{3} \frac{\partial}{\partial \omega} \left[ 2G_{\tilde{P}_{\perp} \tilde{P}_{\parallel}}^R(\mathbf{0}, \omega) + G_{\tilde{P}_{\parallel} \tilde{P}_{\parallel}}^R(\mathbf{0}, \omega) \right] \Big|_{\omega \rightarrow 0}, \\
\eta_0 &= -\frac{1}{4} \frac{\partial}{\partial \omega} \text{Im} G_{\hat{\pi}_{\mu\nu}, \hat{\pi}^{\mu\nu}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\eta_1 &= -\frac{4}{3} \eta_0 + 2 \frac{\partial}{\partial \omega} G_{\tilde{P}_{\parallel} \tilde{P}_{\perp}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\eta_2 &= -\eta_0 + \frac{1}{2} \frac{\partial}{\partial \omega} \text{Im} G_{\hat{R}_{\mu}, \hat{R}^{\mu}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\eta_3 &= \frac{1}{8} \frac{\partial}{\partial \omega} G_{b_{\rho\alpha} \hat{\pi}^{\rho\sigma}, \hat{\pi}_{\sigma}^{\alpha}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\eta_4 &= -\frac{1}{2} \frac{\partial}{\partial \omega} G_{b_{\rho\sigma} \hat{R}^{\rho}, \hat{R}^{\sigma}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0},
\end{aligned} \tag{86}$$

where the retarded Green's function is defined as

$$G_{\hat{A}\hat{B}}^R(\mathbf{x}, t) \equiv -i\theta(t) \left[ \hat{A}(\mathbf{x}, t), \hat{B}(\mathbf{0}, 0) \right].$$

In deriving Eq. (86), we have used the properties  $-\text{Im} G_{\hat{A}\hat{A}}^R(\omega, b^{\sigma}) = \text{Im} G_{\hat{A}\hat{A}}^R(-\omega, b^{\sigma})$  and  $G_{\hat{A}\hat{B}}^R(\omega, b^{\sigma}) = \epsilon_A \epsilon_B G_{\hat{B}\hat{A}}^R(\omega, -b^{\sigma})$  of the retarded Green's function, which are valid for any Hermitian bosonic operators  $\hat{A}$  and  $\hat{B}$ . (Here  $\epsilon_A$  ( $\epsilon_B$ ) is the parity of the operator  $\hat{A}$  ( $\hat{B}$ ) under time-reversal.) These relations are the consequence of the Onsager's reciprocal principle [1]. Similarly, for the thermal conductivities we have

$$\begin{aligned}
\kappa_{\perp} &= \frac{1}{2} \frac{\partial}{\partial \omega} \text{Im} G_{\Xi_{\mu\nu} \hat{G}^{\mu}, \hat{G}^{\nu}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\kappa_{\parallel} &= -\frac{\partial}{\partial \omega} \text{Im} G_{b_{\mu} \hat{G}^{\mu}, b_{\nu} \hat{G}^{\nu}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\kappa_{\times} &= -\frac{1}{2} \frac{\partial}{\partial \omega} G_{b_{\mu\nu} \hat{G}^{\mu}, \hat{G}^{\nu}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}.
\end{aligned} \tag{87}$$

For practical purposes it is sufficient to compute the transport coefficients in the rest frame of the fluid. In this case the correlation functions for the  $\eta$ 's

and  $\kappa$ 's simplify, and we find

$$\begin{aligned}
\eta_0 &= -\frac{\partial}{\partial\omega} \text{Im} G_{\hat{T}^{12}, \hat{T}^{12}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\eta_2 &= -\eta_0 - \frac{\partial}{\partial\omega} \text{Im} G_{\hat{T}^{13}, \hat{T}^{13}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\eta_3 &= -\frac{1}{2} \frac{\partial}{\partial\omega} G_{\hat{P}_\perp, \hat{T}^{12}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\eta_4 &= -\frac{\partial}{\partial\omega} G_{\hat{T}^{13}, \hat{T}^{23}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\kappa_\perp &= -\frac{\partial}{\partial\omega} \text{Im} G_{\hat{G}^1, \hat{G}^1}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\kappa_\parallel &= -\frac{\partial}{\partial\omega} \text{Im} G_{\hat{G}^3, \hat{G}^3}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\kappa_\times &= -\frac{\partial}{\partial\omega} G_{\hat{G}^1, \hat{G}^2}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}.
\end{aligned} \tag{88}$$

Equations (84)-(88) are our main results. They express the transport coefficients of strongly magnetized fluids in terms of equilibrium (retarded) correlation functions, which can be computed by methods of equilibrium statistical mechanics. They take into account the fact that the magnetic field breaks the translational symmetry of the problem, thereby increasing the number of transport coefficients compared to the nonmagnetic case. These expressions are valid at large field strengths (but below values at which vacuum quantum fluctuations become important) and can be used to compute transport coefficients in quantizing fields. A specific example was given in our earlier work [22].

It is instructive to compare our results to those for an isotropic, unmagnetized ( $B = 0$ ) fluid. In the isotropic case, the most general tensor decompositions for the viscosity tensor and thermal conductivity tensor are, respectively,

$$\begin{aligned}
\eta^{\mu\nu\alpha\beta} &= \eta \left( \Delta^{\mu\alpha} \Delta^{\nu\beta} + \Delta^{\mu\beta} \Delta^{\nu\alpha} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta} \right) + \zeta \Delta^{\mu\nu} \Delta^{\alpha\beta}, \\
\kappa^{\mu\nu} &= \kappa \Delta^{\mu\nu}.
\end{aligned} \tag{89}$$

The corresponding Kubo formulas are

$$\zeta = -\frac{\partial}{\partial\omega} \text{Im} G_{\hat{P}\hat{P}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0},$$

$$\begin{aligned}
\eta &= -\frac{1}{10} \frac{\partial}{\partial \omega} \text{Im} G_{\hat{\tau}^{\mu\nu}, \hat{\tau}_{\mu\nu}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\kappa &= \frac{1}{3} \frac{\partial}{\partial \omega} \text{Im} G_{\hat{G}^\mu, \hat{G}_\mu}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0},
\end{aligned} \tag{90}$$

or when written in the rest frame,

$$\begin{aligned}
\zeta &= -\frac{\partial}{\partial \omega} \text{Im} G_{\tilde{P}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\eta &= -\frac{\partial}{\partial \omega} \text{Im} G_{\hat{T}^{12}, \hat{T}^{12}}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0}, \\
\kappa &= -\frac{\partial}{\partial \omega} \text{Im} G_{\hat{G}^1, \hat{G}^1}^R(\mathbf{0}, \omega) \Big|_{\omega \rightarrow 0},
\end{aligned} \tag{91}$$

where

$$\begin{aligned}
\hat{\tau}^{\mu\nu} &\equiv \left( \Delta_{\mu\rho} \Delta_{\nu\sigma} - \frac{1}{3} \Delta_{\mu\nu} \Delta_{\rho\sigma} \right) \hat{T}^{\rho\sigma}, \\
\tilde{P} &\equiv \hat{P} - \left( \frac{\partial P}{\partial \varepsilon} \right)_n \hat{\varepsilon} - \left( \frac{\partial P}{\partial n} \right)_\varepsilon \hat{n}, \\
\hat{P} &\equiv \frac{1}{3} \sum_i \hat{T}^{ii}.
\end{aligned} \tag{92}$$

These expressions are obtained directly from Eq. (86) and Eq. (88) by taking the limit  $B = 0$ . However, it is to be noted that even in the isotropic case one has  $(\tilde{P}_\perp, \tilde{P}_\parallel) \neq (\tilde{P}, \tilde{P})$  for pressure-pressure correlation functions.

Thus, we have succeeded in expressing the transport coefficients of a strongly magnetized fluid in terms of certain correlation functions of quantities appearing in the energy-momentum tensor. For practical evaluation of these correlation functions, one needs to specify a suitable Lagrangian describing the system and a many-body scheme which allows for resummation of polarization-type diagrams in a controlled manner. Typically, non-perturbative resummation schemes are needed to obtain correct infrared behavior of the transport coefficients. Concrete applications have been presented, for example, in Refs. [32, 33].

#### 4. Summary and Discussion

In this work we have extended the development of covariant magneto-hydrodynamics of relativistic fluids initiated in Ref. [22]. A hydrodynamical

description of physical processes in highly magnetized compact objects can be achieved starting from conservation laws for conserved charges and a suitable expansion scheme that systematically resolves the large-length-scale and low-frequency limits. In the most general case, the dissipative function contains five shear viscosities, two bulk viscosities, and three thermal conductivity coefficients, which encode the transport processes at the microscopic level. By utilizing Zubarev’s method of non-equilibrium statistical operators, we have related these transport coefficients to correlation functions of quantities entering the energy-momentum tensor in equilibrium. This is done to linear order in the expansion of non-equilibrium statistical operators with respect to the gradients of relevant statistical parameters (temperature, chemical potentials, and momentum.) The linear perturbation theory recovers the ordinary dissipative Navier-Stokes theory. In general, in order to obtain stable and causal hydrodynamics one needs to expand to second order with respect to the gradients, in which case additional transport coefficients arise. It would be an interesting task to compute these coefficients in the magnetohydrodynamic theory in the limit of large magnetic fields.

We have shown how to relate the correlation function, constructed from ingredients of the energy-momentum tensor, to the retarded Green’s functions of equilibrium theory at vanishing momentum. The static coefficients are obtained, as usual, by taking the limit of vanishing frequency. In order to obtain an explicit form for the transport coefficients, the energy-momentum tensor (or, equivalently, the Lagrangian) describing the system needs to be specified. The two-particle response functions computed at any finite order in the perturbation theory do not give the correct dependence of the transport coefficients on the coupling(s) of the underlying theory. Therefore, a specific resummation scheme is needed in each case. The Kubo formulas obtained in this work provide the necessary starting point for implementing such a program for various systems affected by strong magnetic fields.

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### Appendix A. Dependence of $b^{\mu\alpha}b^{\nu\beta} + b^{\mu\beta}b^{\nu\alpha}$ on combinations in Eq. (32)

In a previous paper [22], we have treated the combination  $b^{\mu\alpha}b^{\nu\beta} + b^{\mu\beta}b^{\nu\alpha}$  as an independent tensor structure, since it has the symmetries of the viscous tensor  $\eta^{\mu\nu\alpha\beta}$ . Here we show that, in fact, it can be written as a linear combination of the tensors appearing in Eq. (32). By direct calculation we find

$$\begin{aligned}
b_{\mu\alpha}b_{\nu\beta} &= b^\lambda u^\xi b^\rho u^\sigma \epsilon_{\mu\alpha\lambda\xi} \epsilon_{\nu\beta\rho\sigma} = -b^\lambda u^\xi b^\rho u^\sigma \begin{vmatrix} g_{\mu\nu} & g_{\mu\beta} & g_{\mu\rho} & g_{\alpha\sigma} \\ g_{\alpha\nu} & g_{\alpha\beta} & g_{\alpha\rho} & g_{\alpha\sigma} \\ g_{\lambda\nu} & g_{\lambda\beta} & g_{\lambda\rho} & g_{\lambda\sigma} \\ g_{\xi\nu} & g_{\xi\beta} & g_{\xi\rho} & g_{\xi\sigma} \end{vmatrix} \\
&= g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\nu\alpha} + b_\beta (b_\alpha g_{\mu\nu} - b_\mu g_{\nu\alpha}) - u_\beta (u_\alpha g_{\mu\nu} - u_\mu g_{\nu\alpha}) \\
&\quad - b_\nu (b_\alpha g_{\mu\beta} - b_\mu g_{\alpha\beta}) + u_\nu (u_\alpha g_{\mu\beta} - u_\mu g_{\alpha\beta}) \\
&\quad - (b_\mu u_\alpha - b_\alpha u_\mu) (b_\nu u_\beta - b_\beta u_\nu) \\
&= \Delta_{\mu\nu}\Delta_{\alpha\beta} - \Delta_{\nu\alpha}\Delta_{\mu\beta} + b_\alpha b_\beta \Delta_{\mu\nu} + b_\mu b_\nu \Delta_{\alpha\beta} - b_\mu b_\beta \Delta_{\nu\alpha} - b_\nu b_\alpha \Delta_{\mu\beta}.
\end{aligned} \tag{A.1}$$

Hence we have

$$\begin{aligned}
b_{\mu\alpha}b_{\nu\beta} + b_{\nu\alpha}b_{\mu\beta} &= 2\Delta_{\mu\nu}\Delta_{\alpha\beta} - \Delta_{\nu\alpha}\Delta_{\mu\beta} - \Delta_{\mu\alpha}\Delta_{\nu\beta} + 2b_\alpha b_\beta \Delta_{\mu\nu} + 2b_\mu b_\nu \Delta_{\alpha\beta} \\
&\quad - b_\mu b_\beta \Delta_{\nu\alpha} - b_\nu b_\alpha \Delta_{\mu\beta} - b_\nu b_\beta \Delta_{\mu\alpha} - b_\mu b_\alpha \Delta_{\nu\beta} \\
&= 2(\text{i}) - (\text{ii}) + 2(\text{iii}) - (\text{v}).
\end{aligned} \tag{A.2}$$

### Appendix B. Derivation of Eq. (71)

The purpose of this Appendix is to give the details of the derivation of Eq. (71). We first establish the following lemmas.

**Lemma 1:** The derivatives of  $\beta = 1/T$  and  $\alpha = \beta\mu$  in thermal equilibrium can be written as

$$\begin{aligned}
D\beta &= \beta (\Theta_\beta \theta + \Phi_\beta \phi), \\
D\alpha &= -\beta (\Theta_\alpha \theta + \Phi_\alpha \phi),
\end{aligned} \tag{B.1}$$



where

$$\begin{aligned}\Theta_\beta &\equiv \left(\frac{\partial P}{\partial \varepsilon}\right)_{n,B}, & \Phi_\beta &\equiv -B \left(\frac{\partial M}{\partial \varepsilon}\right)_{n,B}, \\ \Theta_\alpha &\equiv \left(\frac{\partial P}{\partial n}\right)_{\varepsilon,B}, & \Phi_\alpha &\equiv -B \left(\frac{\partial M}{\partial n}\right)_{\varepsilon,B}.\end{aligned}\quad (\text{B.2})$$

**Proof:** First, using  $\beta d\mu = d\alpha - \mu d\beta$ ,  $d\beta = -\beta^2 dT$ , and  $\varepsilon + P = Ts + \mu n$  we have

$$\begin{aligned}dP &= sdT + nd\mu + MdB \\ &= -T(\varepsilon + P)d\beta + Tnd\alpha + MdB,\end{aligned}\quad (\text{B.3})$$

or, equivalently,

$$\left(\frac{\partial P}{\partial \beta}\right)_{\alpha,B} = -T(\varepsilon + P), \quad \left(\frac{\partial P}{\partial \alpha}\right)_{\beta,B} = Tn, \quad \left(\frac{\partial P}{\partial B}\right)_{\alpha,\beta} = M. \quad (\text{B.4})$$

Secondly, using  $ds = \beta d\varepsilon - \alpha dn + \beta MdB$ , we obtain

$$\begin{aligned}\left(\frac{\partial \beta}{\partial n}\right)_{\varepsilon,B} &= -\left(\frac{\partial \alpha}{\partial \varepsilon}\right)_{n,B}, & \left(\frac{\partial \beta}{\partial B}\right)_{\varepsilon,n} &= \left(\frac{\partial(\beta M)}{\partial \varepsilon}\right)_{n,B}, \\ \left(\frac{\partial \alpha}{\partial B}\right)_{\varepsilon,n} &= -\left(\frac{\partial(\beta M)}{\partial n}\right)_{\varepsilon,B}.\end{aligned}\quad (\text{B.5})$$

Based on the above thermodynamic relations, we may now write

$$\begin{aligned}D\beta &= \left(\frac{\partial \beta}{\partial \varepsilon}\right)_{n,B} D\varepsilon + \left(\frac{\partial \beta}{\partial n}\right)_{\varepsilon,B} Dn + \left(\frac{\partial \beta}{\partial B}\right)_{\varepsilon,n} DB \\ &= -\left(\frac{\partial \beta}{\partial \varepsilon}\right)_{n,B} (\varepsilon + P)\theta - \left(\frac{\partial \beta}{\partial n}\right)_{\varepsilon,B} n\theta \\ &\quad - \left(\frac{\partial \beta}{\partial \varepsilon}\right)_{n,B} MDB + \left(\frac{\partial \beta}{\partial B}\right)_{\varepsilon,n} DB \\ &= \left(\frac{\partial \beta}{\partial \varepsilon} \frac{\partial P}{\partial \beta} - \frac{\partial \beta}{\partial n} \frac{\partial P}{\partial \alpha}\right)_B \beta\theta \\ &\quad - \left(\frac{\partial \beta}{\partial \varepsilon}\right)_{n,B} MDB + \left(\frac{\partial \beta}{\partial B}\right)_{\varepsilon,n} DB \\ &= \beta \left(\frac{\partial P}{\partial \varepsilon}\right)_{n,B} \theta + \beta \left(\frac{\partial M}{\partial \varepsilon}\right)_{n,B} DB,\end{aligned}\quad (\text{B.6})$$

$$\begin{aligned}
D\alpha &= \left(\frac{\partial\alpha}{\partial\varepsilon}\right)_{n,B} D\varepsilon + \left(\frac{\partial\alpha}{\partial n}\right)_{\varepsilon,B} Dn + \left(\frac{\partial\alpha}{\partial B}\right)_{\varepsilon,n} DB \\
&= -\left(\frac{\partial\alpha}{\partial\varepsilon}\right)_{n,B} (\varepsilon + P)\theta - \left(\frac{\partial\alpha}{\partial n}\right)_{\varepsilon,B} n\theta \\
&\quad - \left(\frac{\partial\alpha}{\partial\varepsilon}\right)_{n,B} MDB + \left(\frac{\partial\alpha}{\partial B}\right)_{\varepsilon,n} DB \\
&= \left(\frac{\partial\alpha}{\partial\varepsilon}\frac{\partial P}{\partial\beta} - \frac{\partial\alpha}{\partial n}\frac{\partial P}{\partial\alpha}\right)_B \beta\theta - \left(\frac{\partial\alpha}{\partial\varepsilon}\right)_{n,B} MDB + \left(\frac{\partial\alpha}{\partial B}\right)_{\varepsilon,n} DB \\
&= -\beta\left(\frac{\partial P}{\partial n}\right)_{\varepsilon,B} \theta - \beta\left(\frac{\partial M}{\partial n}\right)_{\varepsilon,B} DB.
\end{aligned} \tag{B.7}$$

From these equations, we immediately recover Eqs. (B.1).

**Lemma 2:** If the electric field is neglected, we have

$$b^\nu \partial_\nu \alpha = \frac{\varepsilon + P}{nT^2} (Tb_\nu Du^\nu - b^\nu \partial_\nu T), \tag{B.8}$$

$$\Xi^{\rho\nu} \partial_\nu \alpha = \frac{\varepsilon + P}{nT^2} (T\Xi^{\rho\nu} Du_\nu - \Xi^{\rho\nu} \partial_\nu T), \tag{B.9}$$

or equivalently,

$$\nabla_\mu \alpha = \frac{\varepsilon + P}{nT^2} (TDu_\mu - \nabla_\mu T). \tag{B.10}$$

**Proof:** When the electric field is neglected,  $E^\mu = 0$ , we have  $\partial_\mu T_{\text{EM}}^{\mu\nu} = -n_\mu F^{\mu\nu} = nE^\nu = 0$ . Substituting  $T_{\text{EM}}^{\mu\nu} = \frac{1}{2}B^2(u^\mu u^\nu - \Xi^{\mu\nu} - b^\mu b^\nu)$  for the energy-momentum tensor of the electromagnetic field, a direct calculation shows that

$$\Xi_{\mu\nu} (\partial^\nu \ln B - Du^\nu + b^\rho \partial_\rho b^\nu) = 0. \tag{B.11}$$

From the first Maxwell equation,  $\partial_\nu (B^\mu u^\nu - B^\nu u^\mu) = 0$ , we have

$$u_\mu \partial_\nu (B^\mu u^\nu - B^\nu u^\mu) = 0 = u^\mu u^\nu \partial_\nu B_\mu - \partial_\nu B^\nu, \tag{B.12}$$

which implies

$$\partial_\nu b^\nu = -b^\nu \partial_\nu \ln B - b^\mu Du_\mu. \tag{B.13}$$

Contracting the second Maxwell equation,  $\partial_\mu H^{\mu\nu} = n^\nu$ , with  $b^{\rho\nu}$ , we obtain

$$\begin{aligned}
0 &= b^{\rho\nu} b_{\mu\nu} \partial^\mu H + H b^{\rho\nu} \partial^\mu b_{\mu\nu} \\
&= \Xi^{\rho\mu} \partial_\mu H + H \epsilon^{\rho\nu\lambda\sigma} \epsilon_{\mu\nu\alpha\beta} b_\lambda u_\sigma (u^\beta \partial^\mu b^\alpha + b^\alpha \partial^\mu u^\beta) \\
&= \Xi^{\rho\mu} \partial_\mu H + H (b^\mu \partial_\mu b^\rho + u^\rho b^\sigma b^\mu \partial_\mu u_\sigma - Du^\rho - b^\rho b^\lambda Du_\lambda) \\
&= \Xi^{\rho\mu} \partial_\mu H + H \Xi^{\rho\mu} (b^\nu \partial_\nu b_\mu - Du_\mu), \tag{B.14}
\end{aligned}$$

which implies

$$\Xi_{\nu\rho} b^\mu \partial_\mu b^\rho = \Xi_{\nu\rho} Du^\rho - \Xi_{\nu\mu} \partial^\mu \ln H. \tag{B.15}$$

Substituting Eq. (B.11) into Eq. (B.15), we obtain

$$\Xi_{\nu\rho} b^\mu \partial_\mu b^\rho = \Xi_{\nu\rho} (Du^\rho - \partial^\rho \ln M). \tag{B.16}$$

The equation of motion of an ideal fluid is

$$\begin{aligned}
0 &= \partial_\mu T_0^{\mu\nu} = \partial_\mu T_{F0}^{\mu\nu} \\
&= u^\nu D\varepsilon + (\varepsilon + P_\perp) u^\nu \theta + (\varepsilon + P_\perp) Du^\nu \\
&\quad - \Xi^{\nu\mu} \partial_\mu P_\perp + MB(b^\nu \partial_\mu b^\mu + b^\mu \partial_\mu b^\nu) + b^\mu b^\nu \partial_\mu P_\parallel. \tag{B.17}
\end{aligned}$$

Contracting this equation with  $b_\nu$  and  $\Xi_{\rho\nu}$  yields

$$(\varepsilon + P_\perp) b_\nu Du^\nu - MB \partial_\mu b^\mu - b^\mu \partial_\mu P_\parallel = 0, \tag{B.18}$$

$$(\varepsilon + P_\perp) \Xi_{\rho\nu} Du^\nu - \Xi_{\rho\mu} \partial^\mu P_\perp + MB \Xi_{\rho\nu} b^\mu \partial_\mu b^\nu = 0. \tag{B.19}$$

Substitution of Eq. (B.13) into Eq. (B.18) followed by some manipulation gives

$$(\varepsilon + P_\perp) b_\nu Du^\nu + MB(b^\nu \partial_\nu \ln B + b^\nu Du_\nu) - b^\mu (s \partial_\mu T + n \partial_\mu \mu + M \partial_\mu B) = 0.$$

We then have

$$nb^\nu \partial_\nu \mu = (\mu n + Ts) b^\nu Du_\nu - sb^\nu \partial_\nu T, \tag{B.20}$$

or equivalently,

$$b^\nu \partial_\nu \alpha = \frac{\varepsilon + P}{nT^2} (T b_\nu Du^\nu - b^\nu \partial_\nu T), \tag{B.21}$$

thereupon proving Eq. (B.8). Substituting Eq. (B.16) into Eq. (B.19), we obtain

$$(\varepsilon + P)\Xi_{\rho\nu}Du^\nu = \Xi_{\rho\nu}(s\partial^\nu T + n\partial^\nu\mu), \quad (\text{B.22})$$

or equivalently,

$$\Xi^{\rho\nu}\partial_\nu\alpha = \frac{\varepsilon + P}{nT^2}(T\Xi^{\rho\nu}Du_\nu - \Xi^{\rho\nu}\partial_\nu T), \quad (\text{B.23})$$

which proves Eq. (B.9).

Making use of Lemmas 1 and 2, we may now write

$$\begin{aligned} \partial_\mu\beta_\nu &= \beta\partial_\mu u_\nu + u_\nu\partial_\mu\beta \\ &= \beta u_\mu\Delta_{\nu\rho}Du^\rho + \beta\nabla_\mu u_\nu + u_\mu u_\nu D\beta + u_\nu\Delta_{\mu\rho}\partial^\rho\beta \\ &= \beta^2(Tu_\mu\Delta_{\nu\rho}Du^\rho - u_\nu\Delta_{\mu\rho}\partial^\rho T) \\ &\quad + \beta(\Delta_{\mu\rho}\Delta_{\nu\sigma} + \Theta_\beta u_\mu u_\nu\Delta_{\rho\sigma} + \Phi_\beta u_\mu u_\nu\Xi_{\rho\sigma})\partial^\rho u^\sigma, \end{aligned} \quad (\text{B.24})$$

$$\begin{aligned} \partial_\mu\alpha &= \Delta_{\mu\nu}\partial^\nu\alpha + u_\mu D\alpha = \frac{\varepsilon + P}{nT^2}(T\Delta_{\mu\nu}Du^\nu - \Delta^{\mu\nu}\partial^\nu T) \\ &\quad - \beta u_\mu(\Theta_\alpha\Delta_{\rho\sigma} + \Phi_\alpha\Xi_{\rho\sigma})\partial^\rho u^\sigma. \end{aligned} \quad (\text{B.25})$$

Substituting these relations into

$$\hat{C} \equiv \hat{T}_{\mu\nu}(\mathbf{x}, t)\partial^\mu\beta^\nu(\mathbf{x}, t) - \hat{N}^\mu(\mathbf{x}, t)\partial_\mu\alpha(\mathbf{x}, t), \quad (\text{B.26})$$

we immediately obtain

$$\begin{aligned} \hat{C} &= \beta^2\left(u_\mu\hat{T}^{\mu\nu}\Delta_{\nu\rho} - \frac{\varepsilon + P}{n}\hat{N}^\nu\Delta_{\nu\rho}\right)(TDu^\rho - \nabla^\rho T) \\ &\quad + \beta\left[\hat{T}^{\mu\nu}(\Delta_{\mu\rho}\Delta_{\nu\sigma} + \Theta_\beta u_\mu u_\nu\Delta_{\rho\sigma} + \Phi_\beta u_\mu u_\nu\Xi_{\rho\sigma})\right. \\ &\quad \left.+ \hat{N}^\mu u_\mu(\Theta_\alpha\Delta_{\rho\sigma} + \Phi_\alpha\Xi_{\rho\sigma})\right]\nabla^\rho u^\sigma \\ &= \left(\frac{n}{\varepsilon + P}u_\mu\hat{T}^{\mu\nu}\Delta_{\nu\rho} - \hat{N}^\nu\Delta_{\nu\rho}\right)\nabla^\rho\alpha \\ &\quad + \beta\left[\hat{T}^{\mu\nu}(\Delta_{\mu\rho}\Delta_{\nu\sigma} + \Theta_\beta u_\mu u_\nu\Delta_{\rho\sigma} + \Phi_\beta u_\mu u_\nu\Xi_{\rho\sigma})\right. \\ &\quad \left.+ \hat{N}^\mu u_\mu(\Theta_\alpha\Delta_{\rho\sigma} + \Phi_\alpha\Xi_{\rho\sigma})\right]w^{\rho\sigma}, \end{aligned} \quad (\text{B.27})$$

with  $w^{\rho\sigma} \equiv (\nabla^\rho u^\sigma + \nabla^\sigma u^\rho)/2$ .

## Appendix C. Integration of Kubo Correlator and Retarded Green's Function

In this Appendix, we derive the relation between the space-time integral of the Kubo correlator and the retarded Green's function. This relation was used to derive Eqs. (80)-(81). Let us denote the integral of a general Kubo correlator by

$$I \equiv \int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{\varepsilon(t'-t)} \left( \hat{X}(\mathbf{x}, t), \hat{Y}(\mathbf{x}', t') \right). \quad (\text{C.1})$$

First, we rewrite the Kubo correlator in the following form, assuming that  $\lim_{t' \rightarrow -\infty} \langle \hat{X}(\mathbf{x}, t) \hat{Y}(\mathbf{x}', t') \rangle_l = \langle \hat{X}(\mathbf{x}, t) \rangle_l \langle \hat{Y}(\mathbf{x}', t') \rangle_l$  so that the correlation vanishes at  $t' \rightarrow -\infty$ :

$$\begin{aligned} \left( \hat{X}(\mathbf{x}, t), \hat{Y}(\mathbf{x}', t') \right) &\equiv \int_0^1 d\tau \langle \hat{X}(\mathbf{x}, t) \left[ e^{-\hat{A}\tau} \hat{Y}(\mathbf{x}', t') e^{\hat{A}\tau} - \langle \hat{Y}(\mathbf{x}', t') \rangle_l \right] \rangle_l \\ &= \frac{1}{\beta} \int_0^\beta d\tau \langle \hat{X}(\mathbf{x}, t) \left[ \hat{Y}(\mathbf{x}', t' + i\tau) - \langle \hat{Y}(\mathbf{x}', t' + i\tau) \rangle_l \right] \rangle_l \\ &= \frac{1}{\beta} \int_0^\beta d\tau \int_{-\infty}^{t'} ds \\ &\quad \frac{d}{ds} \langle \hat{X}(\mathbf{x}, t) \left[ \hat{Y}(\mathbf{x}', s + i\tau) - \langle \hat{Y}(\mathbf{x}', s + i\tau) \rangle_l \right] \rangle_l \\ &= \frac{i}{\beta} \int_{-\infty}^{t'} ds \langle \left[ \hat{X}(\mathbf{x}, t), \hat{Y}(\mathbf{x}', s) \right] \rangle_l. \end{aligned} \quad (\text{C.2})$$

In order to obtain the last equality we have used the Kubo-Martin-Schwinger relation  $\langle \hat{X}(t) \hat{Y}(t' + i\beta) \rangle_l = \langle \hat{Y}(t') \hat{X}(t) \rangle_l$ .

Then we find that

$$\begin{aligned} I &= \int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{\varepsilon(t'-t)} \frac{i}{\beta} \int_{-\infty}^{t'} ds \langle \left[ \hat{X}(\mathbf{x}, t), \hat{Y}(\mathbf{x}', s) \right] \rangle_l \\ &= - \int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{\varepsilon(t'-t)} \frac{1}{\beta} \int_{-\infty}^{t'} ds G_R(\mathbf{x} - \mathbf{x}', t - s) \\ &= - \int_{-\infty}^0 dt' e^{\varepsilon t'} \frac{1}{\beta} \int_{-\infty}^{t'} ds \int \frac{d\omega}{2\pi} e^{i\omega s} \lim_{\mathbf{k} \rightarrow 0} G_R(\mathbf{k}, \omega) \\ &= - \frac{1}{\beta} \int \frac{d\omega}{2\pi} \frac{1}{i\omega(i\omega + \varepsilon)} \lim_{\mathbf{k} \rightarrow 0} G_R(\mathbf{k}, \omega) \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\beta} \oint \frac{d\omega}{2\pi} \frac{1}{i\omega(i\omega + \varepsilon)} \lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \left[ G_R(\mathbf{k}, 0) + \omega \frac{\partial}{\partial \omega} G_R(\mathbf{k}, \omega) + \dots \right] \\
&= \frac{i}{\beta} \lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \frac{\partial}{\partial \omega} G_R(\mathbf{k}, \omega),
\end{aligned} \tag{C.3}$$

the contour integration being performed in the upper-half plane where the retarded Green's function is analytic. If  $\hat{A} = \hat{B}$ , the real (imaginary) part of the retarded Green's function is an even (odd) function of  $\omega$ ,

$$I = -\frac{1}{\beta} \lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \frac{\partial}{\partial \omega} \text{Im} G_R(\mathbf{k}, \omega). \tag{C.4}$$

This result is used in Sec. 3 to obtain Eqs. (80)-(81).

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